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# Conservation properties of vectorial operator splitting

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## Abstract

This work is concerned with the conservation properties of a new vectorial operator splitting scheme for solving the incompressible Navier–Stokes equations. It is proven that the difference approximation of the advection operator conserves square of velocity components and the kinetic energy as the differential operator does, while pressure term conserves only the kinetic energy. Some analytical requirements necessary to be satisfied of difference schemes for incompressible Navier–Stokes equations are formulated and discussed. The properties of the methods are illustrated with results from numerical computations for lid-driven cavity flow. © 2002 Elsevier Science B.V. All rights reserved.

**Keywords:** Incompressible Navier–Stokes; Conservation properties; Stability and convergence of difference schemes

## 1. Introduction

In this work, we examine the conservation properties of a new scheme proposed in [3,4,10] for solving incompressible Navier–Stokes equations. The most important problem is how to construct convergent difference scheme. Since the convergence is a consequence of consistency and stability thus it is necessary to choose those approximating schemes that are stable. It is naturally to have stability in the norms of the spaces for which the original problem is stable. For the well-posed

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problems of mathematical physics these are the energy spaces where the squares of the norms express the total energy of the systems. Because of this, we have to analyze the derivation of the energy estimations in the differential case and to construct the scheme for which we can satisfy this derivation in the corresponding Hilbert space in the discrete case. However, the criteria of consistency and stability become complicated when applied to the solution of nonlinear partial differential equations. Therefore, the difference scheme has to be conservative, namely, its conservation laws to be satisfied identically. Then the nonlinearity is not invincible task.

According to [12] the conservation properties of the mass, momentum, and kinetic energy equations for incompressible flow are specified as analytical requirements for a proper set of discrete equations. In addition, in present work we summarize some of the analytical requirements necessary to be satisfied of the difference scheme. In such way one can improve the convergence of the numerical solutions for the incompressible Navier–Stokes equations. Our objective is to examine the vectorial operator splitting numerical scheme for its conservation properties and other requirements.

The article is organized as follows. The problem is formulated in Section 2. The vectorial operator-splitting method for solving the incompressible Navier–Stokes equations is presented in Section 3. The analytical requirements and the difference approximations of the operators with discussion of their conservation properties are given in Section 4. Numerical results are presented and discussed in Section 5. Finally, in Section 6 some conclusions are drawn.

## 2. Incompressible Navier–Stokes equations

Consider the momentum equation

$$\frac{\partial \mathbf{u}}{\partial t} + C[\mathbf{u}] + P[\mathbf{u}] - V[\mathbf{u}] = 0 \quad (1)$$

and the continuity equation

$$\nabla \cdot \mathbf{u} = 0 \quad (2)$$

in a closed domain  $\Omega$  with a piecewise smooth boundary  $\partial\Omega$ . Here  $\mathbf{x} = (x, y, z) \in \Omega$  are Cartesian coordinates,  $\mathbf{u} = \mathbf{u}(\mathbf{x})$  is the velocity vector,

$$P[\mathbf{u}] = (P_x[u], P_y[v], P_z[w]) = \nabla p = \left( \frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z} \right),$$

where  $p = p(\mathbf{x}, t)$  is the pressure. In Eq. (1) the operator  $C = C_x + C_y + C_z$  is a short-hand notation for the advection term. For the viscous term  $V$  we use  $V = V_x + V_y + V_z = 1/Re(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2) = 1/Re\Delta$ . The Reynolds number is defined as  $Re = UL/\nu$ , where  $U$  is the characteristic velocity,  $L$ —characteristic length,  $\nu$ —kinematic coefficient of viscosity.

In our consideration, we assume divergence free initial conditions

$$\mathbf{u}|_{t=0} = \mathbf{u}_0 \quad (3)$$

and boundary conditions

$$\mathbf{u}|_{\partial\Omega} = \mathbf{u}_b, \quad (4)$$

i.e., the velocity is prescribed at the boundary.

Instead of the continuity equation, we use an equation for pressure by modifying the well-known “Poisson equation for pressure” obtained from (1) acknowledging (2)

$$\Delta p = -\nabla \cdot C[\mathbf{u}]. \quad (5)$$

Eq. (5) is balanced by multiplying by  $1/Re$  and using the divergent equation

$$-\frac{1}{Re} \Delta p + \gamma \nabla \cdot \mathbf{u} = \frac{1}{Re} \nabla \cdot C[\mathbf{u}]. \quad (6)$$

Similar form of pressure equation is presented in [7].

The formulation with equation for pressure (5) is equivalent to the original system only if the continuity equation is satisfied also on the boundary [4], i.e., the following boundary condition has to be used as well

$$\nabla \cdot \mathbf{u}|_{\partial\Omega} = 0. \quad (7)$$

Now the problem has exact number of boundary conditions for the equations and it is overposed if velocity is decoupled from the pressure.

After discretizing in time it is necessary to solve a nonlinear steady problem at each time step. We use pseudo-transient approach for the stationary Navier–Stokes equations. It is well known that for the solution of the nonlinear elliptic problems a lot of efficient methods exist. One can solve the resulting algebraic equations in a segregated or a coupled manner. The latter is more efficient in general especially in the case of high Reynolds number flows when the stability is very important.

### 3. Vectorial operator splitting method

The vectorial operator splitting method was proposed in [3] for solving steady incompressible Navier–Stokes equations at large values of Reynolds numbers. The splitting procedure reduces in order of magnitude the number of operations per iteration comparing with application of direct solvers. The latter requires large memories and it is not feasible for large scale computations, particularly, for three-dimensional problems: splitting or multi-grid are very efficient techniques for solution of nonlinear elliptic systems. Despite of other splitting schemes like the alternating direction scheme the present one is applicable for the multi-dimensional case.

#### 3.1. Time discretization

To solve the unsteady problem (1), (2) the fully implicit backward Euler scheme is used here in order to ensure strong stability of the method. One obtains the following problems:

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\tau} + C[\mathbf{u}^{n+1}] + P[\mathbf{u}^{n+1}] - V[\mathbf{u}^{n+1}] = 0, \quad (8)$$

$$-\frac{1}{Re} \Delta p^{n+1} + \gamma \nabla \cdot \mathbf{u}^{n+1} = \frac{1}{Re} \nabla \cdot C[\mathbf{u}^{n+1}], \quad (9)$$

$$\mathbf{u}^{n+1}|_{\partial\Omega} = \mathbf{u}_b, \quad \nabla \cdot \mathbf{u}^{n+1}|_{\partial\Omega} = 0. \quad (10)$$

At each time step one has to solve the following stationary problems for  $\mathbf{u} = \mathbf{u}^{n+1}$  and  $p = p^{n+1}$ :

$$-\frac{1}{Re} \Delta \mathbf{u} + \nabla p + C[\mathbf{u}] + \frac{\mathbf{u}}{\tau} = \frac{\mathbf{u}^n}{\tau}, \quad (11)$$

$$-\frac{1}{Re} \Delta p + \gamma \nabla \cdot \mathbf{u} = \frac{1}{Re} \nabla \cdot C[\mathbf{u}], \quad (12)$$

$$\mathbf{u}|_{\partial\Omega} = \mathbf{u}_b, \quad \nabla \cdot \mathbf{u}|_{\partial\Omega} = 0. \quad (13)$$

As it has been already mentioned, Eqs. (11)–(13) can be solved in a coupled manner. To do this, we use a generalization of the splitting algorithm proposed in [3]. In the case when we are interested only in the solution of the steady problem, it is better to solve it directly. In other words, in order to solve directly the steady problem, it is enough to perform only one step with respect to the real time using  $\tau = \infty$  in Eq. (11).

### 3.2. Pseudo-transient approach to generalized stationary problems

To compute the stationary solution we render system (11), (12) to an evolution system by adding in each equation of the governing system, derivatives with respect to an *artificial* time  $s$ , see [8,13]. Thus, we obtain the following parabolic problem:

$$\frac{\partial \boldsymbol{\theta}}{\partial s} = L[\boldsymbol{\theta}] + N[\boldsymbol{\theta}] + \mathbf{F}[\boldsymbol{\theta}], \quad (14)$$

where

$$\boldsymbol{\theta} = \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix}, \quad L = \begin{pmatrix} \Delta/Re - 1/\tau & -\nabla \\ -\gamma \nabla & \Delta/Re \end{pmatrix},$$

$$N = \begin{pmatrix} -C & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{F}[\boldsymbol{\theta}] = \begin{pmatrix} \mathbf{u}^n/\tau \\ \nabla \cdot C[\mathbf{u}]/Re \end{pmatrix}$$

with boundary conditions given by (13).

In incompressible flows the pressure is defined up to an arbitrary function of time. For the sake of convenience we define this function similarly to [1] as the average of the pressure at the specific time stage, i.e., we assume (for pressure uniqueness) that the following relation is satisfied

$$\int_{\Omega} p(\mathbf{x}, t, s) d\mathbf{x} = 0, \quad t > 0, \quad s > 0. \quad (15)$$

The pseudo-time step is an additional parameter in the scheme that can be varied to accelerate convergence. Upon convergence, the pseudo-time term vanishes, and the steady equations are satisfied. Each iteration (time step) can be implemented via operator splitting because of its computational efficiency.

In spite of that the defined evolution problem (14), (13) is not the real physical problem, its stationary solutions are solutions of equations (11)–(13) (respectively (1), (2) with boundary condition

(4) for fixed  $t > 0$ ). The idea of using the false transient method is in order to construct robust and efficient difference scheme for obtaining stationary solutions of the incompressible Navier–Stokes equations.

### 3.3. Operator splitting for generalized stationary problems

We employ a generalization of the scheme of Douglas and Rachford [6]. One of the main reasons for our choice to make use of the this scheme is that it can be applied in the three-dimensional case as well. For splitting the operator  $A = A_1 + \dots + A_l$  in the equation

$$\frac{\partial \theta}{\partial t} = A\theta + G \quad (16)$$

we make the following steps:

$$\frac{\theta^{m+1/l} - \theta^m}{\sigma} = A_1 \theta^{m+1/l} + \sum_{i=2}^l A_i \theta^m + G^m, \quad (17)$$

$$\frac{\theta^{m+i/l} - \theta^{m+(i-1)/l}}{\sigma} = A_i (\theta^{m+i/l} - \theta^m), \quad i = 2, \dots, l, \quad (18)$$

where  $l=2$  and  $3$  in two and three dimensional case, respectively. The splitting scheme approximates fully implicit backward Euler scheme. We are interested in the converged solution and therefore the order of approximation with respect to the artificial time is not important. Note that the first fractional step produces consistency with the equation, and the next steps are introduced to improve the stability. For this reason the splitting scheme is called a scheme with stabilizing correction, see [14].

After excluding  $\theta^{m+i/l}$ ,  $i = 1, \dots, l-1$ , from (17) to (18) the scheme in whole step takes the form

$$\prod_{i=1}^l (I - \sigma A_i) \frac{\theta^{m+1} - \theta^m}{\sigma} = A\theta^m + G^m. \quad (19)$$

It follows from (19) the complete consistency of the splitting scheme with Eq. (16).

In 2D case Eq. (19) can be written in the following form:

$$(I + \sigma^2 A_1 A_2) \frac{\theta^{m+1} - \theta^m}{\sigma} = (A_1 + A_2) \theta^{m+1} + G^m, \quad (20)$$

while the respective form in 3D is

$$[I + \sigma^2 (A_1 A_2 + A_2 A_3 + A_3 A_1) - \sigma^3 A_1 A_2 A_3] \frac{\theta^{m+1} - \theta^m}{\sigma} = (A_1 + A_2 + A_3) \theta^{m+1} + G^m. \quad (21)$$

We take

$$A_1 = L_x + N_x, \quad A_2 = L_y + N_y, \quad A_3 = L_z + N_z, \quad (22)$$

where  $L_x$ ,  $N_x$  are the respective operators of the derivatives with respect to  $x$ ,  $L_y$ ,  $N_y$ —with respect to  $y$ , and  $L_z$ ,  $N_z$ —with respect to  $z$ -direction.

The above splitting schemes satisfy the desirable property that, if the numerical solution converges, its steady-state solutions are independent of the time step. It is readily seen from (20) and (21) that upon convergence, i.e.,  $\|\boldsymbol{\theta}^{m+1} - \boldsymbol{\theta}^m\| \rightarrow 0$ , the solution of the evolution problem satisfies the stationary problem and does not depend on the artificial time step increment.

The main advantage is that due to the economy of the computer time required of splitting schemes, the schemes of stabilizing corrections are very efficient for solving multi-dimensional problems. The operator-splitting schemes are economical as explicit schemes and can retain the unconditional stability inherent in some of the implicit schemes.

#### 4. Difference problem

In this section, we formulate the conservation properties and requirements of differential operators for a fixed time level  $t > 0$ . After specifying the analytical requirements, we define the grid and difference approximations, and then discuss the conservation properties of the scheme.

##### 4.1. Analytical requirements

At first, let us introduce the Hilbert space  $H(\Omega)$  of vector-functions with scalar product

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_i (\alpha_i, \beta_i), \quad (\alpha_i, \beta_i) = \int_{\Omega} \alpha_i(\mathbf{x}) \beta_i(\mathbf{x}) d\mathbf{x} \quad (23)$$

and the corresponding norm  $\|\boldsymbol{\alpha}\| = (\boldsymbol{\alpha}, \boldsymbol{\alpha})^{1/2}$ , which will be used later.

We summarize the following analytical requirements necessary to be satisfied of the difference scheme:

(i) *Conservation properties*: According to [12] we introduce

**Definition 1.** The term  $T(\varphi)$  is conservative if it can be written in divergence form

$$T[\cdot] = \nabla \cdot (S[\cdot]) \quad (24)$$

and it is well known that

- (a) The *mass* is conserved ‘a priori’ since the continuity equation (2) appears in divergence form.
- (b) *Momentum* is conserved ‘a priori’ if the continuity equation (2) is satisfied: pressure and viscous terms are conservative ‘a priori’; the convective term is also conservative ‘a priori’ if  $\nabla \cdot \mathbf{u} = 0$ .
- (c) *Square of a velocity component*  $\varphi^2$ : The advection operator conserves  $\varphi^2$  if a skew-symmetric form is used since for the nonlinear advection operators one has

$$C_x[\varphi] = \frac{1}{2} \left[ \frac{\partial(\varphi u)}{\partial x} + u \frac{\partial \varphi}{\partial x} \right], \quad C_y[\varphi] = \frac{1}{2} \left[ \frac{\partial(\varphi v)}{\partial y} + v \frac{\partial \varphi}{\partial y} \right],$$

$$C_z[\varphi] = \frac{1}{2} \left[ \frac{\partial(\varphi w)}{\partial z} + w \frac{\partial \varphi}{\partial z} \right].$$

Here  $\varphi$  is one of the velocity components  $u$ ,  $v$ , and  $w$ . For instance, in the direction  $x$

$$\varphi C_x[\varphi] = \frac{\varphi}{2} \left[ \frac{\partial(\varphi u)}{\partial x} + u \frac{\partial \varphi}{\partial x} \right] = \frac{1}{2} \frac{\partial(\varphi^2 u)}{\partial x}. \quad (25)$$

Hence, the operator  $C_x$  is conserving the square of a velocity component  $\varphi^2$ . It means that under the assumption of homogenous boundary conditions, we have

$$(C_x[\varphi], \varphi) = (C_y[\varphi], \varphi) = (C_z[\varphi], \varphi) = 0 \quad \text{or} \quad (C[\varphi], \varphi) = 0. \quad (26)$$

The pressure term in the momentum equation is not conservative, since

$$u \frac{\partial p}{\partial x} = \frac{\partial(Up)}{\partial x} - p \frac{\partial u}{\partial x} \quad (27)$$

for the velocity component  $u$ , for instance.

Similarly, for the viscous term in the equation for  $u$  satisfies

$$u \Delta u = u \nabla^2 u = \nabla \cdot (u \nabla u) - (\nabla u)^2 \quad (28)$$

and, in other words, it is not conservative as well.

- (d) *Kinetic energy*  $K \stackrel{\text{def}}{=} \frac{1}{2}(u^2 + v^2 + w^2)$ : It follows from (15) that the operator  $C[\mathbf{u}] = \frac{1}{2}[\nabla \cdot (\mathbf{u}\mathbf{u}) + \mathbf{u} \cdot \nabla \mathbf{u}]$  conserves the kinetic energy  $K$ , i.e.

$$\begin{aligned} \mathbf{u} \cdot C[\mathbf{u}] &= \frac{1}{2} \mathbf{u} \cdot [\nabla \cdot (\mathbf{u}\mathbf{u}) + \mathbf{u} \cdot \nabla \mathbf{u}] \\ &= \frac{1}{2} [\mathbf{u} \cdot \nabla (\mathbf{u}^2) + \mathbf{u}^2 \cdot \nabla \mathbf{u}] = \frac{1}{2} [\nabla \cdot (\mathbf{u}\mathbf{u}^2)]. \end{aligned} \quad (29)$$

The pressure term is energy conservative if the continuity equation is satisfied

$$\mathbf{u} \cdot \nabla p = \nabla \cdot (\mathbf{u}p) - p(\nabla \cdot \mathbf{u}) = \nabla \cdot (\mathbf{u}p), \quad (30)$$

while the viscous term is not conservative

$$\mathbf{u} \cdot \Delta \mathbf{u} = \mathbf{u} \cdot \nabla^2 \mathbf{u} = \nabla \cdot (\mathbf{u} \nabla \mathbf{u}) - (\nabla \mathbf{u})^2 \quad (31)$$

because of the kinetic energy dissipation—the second term in the right-hand side of (31).

- (ii) *Compatibility condition for Poisson equation for pressure* [1,5] should be satisfied if the numerical method uses a Poisson equation for pressure instead of the continuity equation

$$\int_{\Omega} F_p \, d\mathbf{x} = \frac{1}{Re} \oint_{\partial\Omega} \frac{\partial p}{\partial n} \, ds, \quad F_p = \gamma \nabla \cdot \mathbf{u} - \frac{1}{Re} \nabla \cdot C[\mathbf{u}]. \quad (32)$$

In (32)  $F_p$  is the right-hand side for pressure equation,  $n$  is the outward normal to the boundary contour  $\partial\Omega$ .

- (iii) *Commutativity* of Laplacian operator  $\Delta$  and divergence operator  $\nabla$ .  
 (iv) *Consistency* between gradient and divergence operators

$$\int_{\Omega} [\mathbf{u} \cdot \nabla p + p(\nabla \cdot \mathbf{u})] \, d\mathbf{x} = \oint_{\partial\Omega} p v_n \, ds \quad (33)$$

should be satisfied as well. For instance, the consistency is necessary in order to obtain skew-symmetric operator  $P$ . The mutually consistent discretizations of operators gradient and divergence with a first-order truncation error on nonstaggered grids are derived in [5]. In the case of such grids the use of standard central difference approximation for gradient leads to the same approximation for divergence and yields for the discretization of pressure Laplace operator an extended stencil and checker-board effect. On staggered grid the consistency between gradient and divergence operators is not difficult to be satisfied.

- (v) *Solenoidally* of the velocity field at each time step, i.e., the continuity equation (2) must be satisfied for any  $t > 0$ .

The satisfaction of (1)–(5) leads to strong  $L_2$  stability of the scheme. Therefore, the purpose is to derive difference scheme that satisfies the above requirements in a discrete sense.

#### 4.2. Difference operators

We choose the approximations of the differential equations and boundary conditions for which the numerical scheme preserves the integral properties of the respective differential problem. It is not trivial task in construction finite difference schemes especially in the case of operator-splitting.

For the case under consideration the flow occupies the region with rectilinear boundaries in cartesian coordinates. The boundary conditions deriving from the continuity equation (2) in the three-dimensional case read

$$\left. \frac{\partial u}{\partial x} \right|_{(x=c, y, z)} = g_1(y, z), \quad \left. \frac{\partial v}{\partial y} \right|_{(x, y=c, z)} = g_2(x, z), \quad \left. \frac{\partial w}{\partial z} \right|_{(x, y, z=c)} = g_3(x, y), \quad (34)$$

where  $(x = c, y, z)$ ,  $(x, y = c, z)$ , and  $(x, y, z = c)$  are boundary points;  $c$  is a generic constant, which can be different;  $g_i$ ,  $i = 1, 2, 3$  are given functions. We keep the coupling between the pressure and the respective velocity component through the boundary conditions at each fractional time step. It allows us to construct robust implicit splitting scheme with strong  $L_2$ -stability.

The grid is *staggered* for  $u$  in  $x$ -direction, for  $v$  in  $y$ -direction, and for  $w$  in  $z$ -direction. For boundary conditions involving derivatives this allows one to use central differences with second-order of approximation on two-point stencils. The number of main grid lines (which are, in fact, the  $p$ -grid lines) in the  $x$ -,  $y$ - and  $z$ -directions are, respectively,  $N_x$ ,  $N_y$ , and  $N_z$ . The coordinates of the grid points are  $(x_i, y_j, z_k)$  for  $i = 1, \dots, N_x$ ,  $j = 1, \dots, N_y$ ,  $k = 1, \dots, N_z$ . The spacings are given by  $h_{x,i}^p = x_{i+1} - x_i$ ,  $i = 1, \dots, N_x - 1$ ,  $h_{y,j}^p = y_{j+1} - y_j$ ,  $j = 1, \dots, N_y - 1$ , and  $h_{z,k}^p = z_{k+1} - z_k$ ,  $k = 1, \dots, N_z - 1$ . The spacings for the function  $u$  in direction  $x$  are defined as follows:

$$h_{x,1}^u = h_{x,1}^p, \quad h_{x,i}^u = \frac{1}{2}(h_{x,i}^p + h_{x,i-1}^p) \quad \text{for } i = 2, \dots, N_x - 1, \quad \text{and } h_{x,N_x}^u = h_{x,N_x-1}^p.$$

Similarly the spacings for  $v$  in direction  $y$  and for  $w$  in direction  $z$  are defined

$$h_{y,1}^v = h_{y,1}^p, \quad h_{y,j}^v = \frac{1}{2}(h_{y,j}^p + h_{y,j-1}^p) \quad \text{for } j = 2, \dots, N_y - 1, \quad h_{y,N_y}^v = h_{y,N_y-1}^p,$$

$$h_{z,1}^w = h_{z,1}^p, \quad h_{z,k}^w = \frac{1}{2}(h_{z,k}^p + h_{z,k-1}^p) \quad \text{for } k = 2, \dots, N_z - 1, \quad h_{z,N_z}^w = h_{z,N_z-1}^p.$$



The pressure is sampled at the points labelled by “•”; function  $u$ —at “○”; function  $v$ —at “\*”, and function  $w$ —at “◇”. The following notations are used:

$$p_{i,j,k} = p(x_i, y_j, z_k), \quad u_{i,j,k} = u(x_i - \frac{1}{2}h_{x,i-1}^p, y_j, z_k),$$

$$v_{i,j,k} = v(x_i, y_j - \frac{1}{2}h_{y,j-1}^p, z_k), \quad w_{i,j,k} = w(x_i, y_j, z_k - \frac{1}{2}h_{z,k-1}^p).$$

For the second derivatives standard three point difference approximations are employed, which inherit the negative definiteness of the respective differential operators. For instance, in direction  $x$  the approximation has the form

$$\frac{\partial^2 f}{\partial x^2} \Big|_{i,j,k} \approx \frac{2}{h_{x,i}^f + h_{x,i-1}^f} \left( \frac{f_{i+1,j,k} - f_{i,j,k}}{h_{x,i}^f} - \frac{f_{i,j,k} - f_{i-1,j,k}}{h_{x,i-1}^f} \right), \quad (35)$$

where  $f$  stands for one of the functions  $u$ ,  $v$ ,  $w$  or  $p$ .

The first derivatives for pressure at the mesh-point labelled by “○”, “\*”, and “◇” are approximated in the following way:

$$P_x^h[u] \Big|_{\circ} = \frac{p_{i,j,k} - p_{i-1,j,k}}{h_{x,i-1}^p}, \quad P_y^h[v] \Big|_{*} = \frac{p_{i,j,k} - p_{i,j-1,k}}{h_{y,j-1}^p}, \quad P_z^h[w] \Big|_{\diamond} = \frac{p_{i,j,k} - p_{i,j,k-1}}{h_{z,k-1}^p}.$$

On the other hand, the derivatives  $\partial u / \partial x$ ,  $\partial v / \partial y$ , and  $\partial w / \partial z$  in  $\nabla \cdot \mathbf{u}$  at each interior mesh-point labelled by “•” are approximated as

$$\frac{\partial u}{\partial x} \Big|_{\bullet} \approx \frac{u_{i+1,j,k} - u_{i,j,k}}{h_{x,i}^u}, \quad \frac{\partial v}{\partial y} \Big|_{\bullet} \approx \frac{v_{i,j+1,k} - v_{i,j,k}}{h_{y,j}^v}, \quad \frac{\partial w}{\partial z} \Big|_{\bullet} \approx \frac{w_{i,j,k+1} - w_{i,j,k}}{h_{z,k}^w}.$$

The operator  $\nabla \cdot \mathbf{C}[\mathbf{u}]$  is approximated in similar way such as those in [4]. The functions  $u$ ,  $v$ , and  $w$  in the approximation of the operator  $F$  are taken at the “old” time stage.

The skew-symmetric difference approximation of the advection term was proposed by Arakawa [2] for the  $\psi - \omega$  formulation for ideal flows. A similar idea in primitive variables was elaborated in [9] with a special reference to the operator-splitting schemes. In [3,4] we consider second-order conservative approximations of the nonlinear operators on a uniform staggered mesh. On a nonuniform staggered mesh, see [10], we employ the following conservative approximations for the nonlinear terms in the momentum equation for velocity component  $u$

$$C_x^h[u] = \left( \frac{\partial(u^2)}{\partial x} - \frac{u}{2} \frac{\partial u}{\partial x} \right) \Big|_{\circ} = \frac{u_{i+1/2,j,k}^m u_{i+1,j,k} - u_{i-1/2,j,k}^m u_{i-1,j,k}}{h_{x,i}^u + h_{x,i-1}^u}, \quad (36)$$

$$C_y^h[u] = \left( \frac{\partial(uv)}{\partial y} - \frac{u}{2} \frac{\partial v}{\partial y} \right) \Big|_{\circ} = \frac{v_{i-1/2,j+1,k}^m u_{i,j+1,k} - v_{i-1/2,j,k}^m u_{i,j-1,k}}{h_{y,j}^p + h_{y,j-1}^p}, \quad (37)$$

$$C_z^h[u] = \left( \frac{\partial(uw)}{\partial z} - \frac{u}{2} \frac{\partial w}{\partial z} \right) \Big|_{\circ} = \frac{w_{i-1/2,j,k+1}^m u_{i,j,k+1} - w_{i-1/2,j,k}^m u_{i,j,k-1}}{h_{z,k}^p + h_{z,k-1}^p}, \quad (38)$$

where  $u_{i+1/2,j,k}^m = (u_{i+1,j,k}^m + u_{i,j,k}^m)/2$ ,  $u_{i-1/2,j,k}^m = (u_{i,j,k}^m + u_{i-1,j,k}^m)/2$ , etc. The differences for nonlinear terms in the equations for  $v$  and  $w$  are similar to (36)–(38).

### 4.3. Conservation properties of defined difference operators

It can be proven that the defined approximations of the nonlinear advection terms preserve their skew-symmetric property. The following statement is valid

**Lemma 1.** *Let appropriate (homogenous, periodic, etc.) boundary conditions are acknowledged and the scalar product is*

$$(\alpha, \beta) \stackrel{\text{def}}{=} \sum_{i,j,k} \alpha_{i,j,k} \beta_{i,j,k} \hbar_{x,i}^f \hbar_{y,j}^f \hbar_{z,k}^f, \quad (39)$$

where

$$\hbar_{x,i}^f = \frac{h_{x,i}^f + h_{x,i-1}^f}{2}, \quad \hbar_{y,j}^f = \frac{h_{y,j}^f + h_{y,j-1}^f}{2}, \quad \hbar_{z,k}^f = \frac{h_{z,k}^f + h_{z,k-1}^f}{2}$$

and  $f = u, v, w$ , or  $p$ . Then the equalities hold true

$$(C_x^h[u], u) = 0, \quad (C_y^h[u], u) = 0, \quad (C_z^h[u], u) = 0, \quad (40)$$

$$(C_x^h[v], v) = 0, \quad (C_y^h[v], v) = 0, \quad (C_z^h[v], v) = 0, \quad (41)$$

$$(C_x^h[w], w) = 0, \quad (C_y^h[w], w) = 0, \quad (C_z^h[w], w) = 0. \quad (42)$$

**Proof.** It is enough to prove the first equality, namely  $(C_x^h[u], u) = 0$ . From the definition of  $C_x^h[u]$ , see (36), it follows that

$$\begin{aligned} (C_x^h[u], u) &= \sum_{i,j,k} u_{i,j,k} \frac{u_{i+1/2,j,k}^m u_{i+1,j,k} - u_{i-1/2,j,k}^m u_{i-1,j,k}}{h_{x,i}^u + h_{x,i-1}^u} \hbar_{x,i}^u \hbar_{y,j}^p \hbar_{z,k}^p \\ &= \frac{1}{2} \sum_{i,j,k} u_{i,j,k} (u_{i+1/2,j,k}^m u_{i+1,j,k} - u_{i-1/2,j,k}^m u_{i-1,j,k}) \hbar_{y,j}^p \hbar_{z,k}^p \\ &= \frac{1}{2} \sum_{j,k} [u_{2,j,k} (u_{3/2,j,k}^m u_{3,j,k} - u_{1/2,j,k}^m u_{1,j,k}) \\ &\quad + u_{3,j,k} (u_{5/2,j,k}^m u_{4,j,k} - u_{3/2,j,k}^m u_{2,j,k}) \\ &\quad + \cdots + u_{N_x-1,j,k} (u_{N_x-1/2,j,k}^m u_{N_x,j,k} - u_{N_x-3/2,j,k}^m u_{N_x-2,j,k}) \\ &\quad + u_{N_x,j,k} (u_{N_x+1/2,j,k}^m u_{N_x+1,j,k} - u_{N_x-1/2,j,k}^m u_{N_x-1,j,k})] \hbar_{y,j}^p \hbar_{z,k}^p \\ &= \frac{1}{2} \sum_{j,k} [u_{N_x,j,k} u_{N_x+1/2,j,k}^m u_{N_x+1,j,k} - u_{2,j,k} u_{1/2,j,k}^m u_{1,j,k}] \hbar_{y,j}^p \hbar_{z,k}^p. \end{aligned}$$

If we assume homogenous boundary conditions, i.e., for the velocity component  $u$  is valid  $u|_{\partial\Omega} = 0$ , then the difference approximation of this boundary condition is  $u_{1/2,j,k} = u_{N_x+1/2,j,k} = 0$ . It follows immediately that  $(C_x^h[u], u) = 0$ .  $\square$

The rest equalities can be proven in a similar way. Hence, the defined approximations of the nonlinear terms on a nonuniform staggered grid preserve their skew-symmetric property. It follows immediately that:

**Theorem 1.** *Under the assumptions of Lemma 1, the following relations are satisfied*

$$(C^h[u], u) = (C^h[v], v) = (C^h[w], w) = 0. \quad (43)$$

From the above theorem it follows

$$(C^h[u], u) + (C^h[v], v) + (C^h[w], w) = 0, \quad (44)$$

hence

**Corollary 1.** *The advection term is energy conservative.*

Similarly, it is not difficult to be proven (taking into account the approximation of the divergence operator) that the pressure term approximation conserves the kinetic energy  $K$  in the case of uniform grids. Under the assumptions of Lemma 1 the following relation is satisfied

$$(P_x^h[u], u) + (P_y^h[v], v) + (P_z^h[w], w) = 0 \quad (45)$$

and the result can be summarized in the next

**Theorem 2.** *The pressure term is energy conservative if the grid is uniform.*

## 5. Numerical results and discussion

It is verified that the proposed numerical scheme satisfies the formulated in Section 4.1 requirements by various computational experiments using the well-known lid-driven cavity problem (the flow is driven by the upper wall with a constant velocity  $U = 1$ ).

### 5.1. Algorithm for stationary problems

At first the algorithm for solving the stationary problem is tested in order to confirm that the theorems and lemmas in Section 4.3 are essentially valid. Therefore, we performed calculations with steady problem to ascertain that the equalities (40)–(42) and (45) are satisfied. The difference equivalence of the requirements (32), (33) and (2) are also verified numerically. Both the 2D and

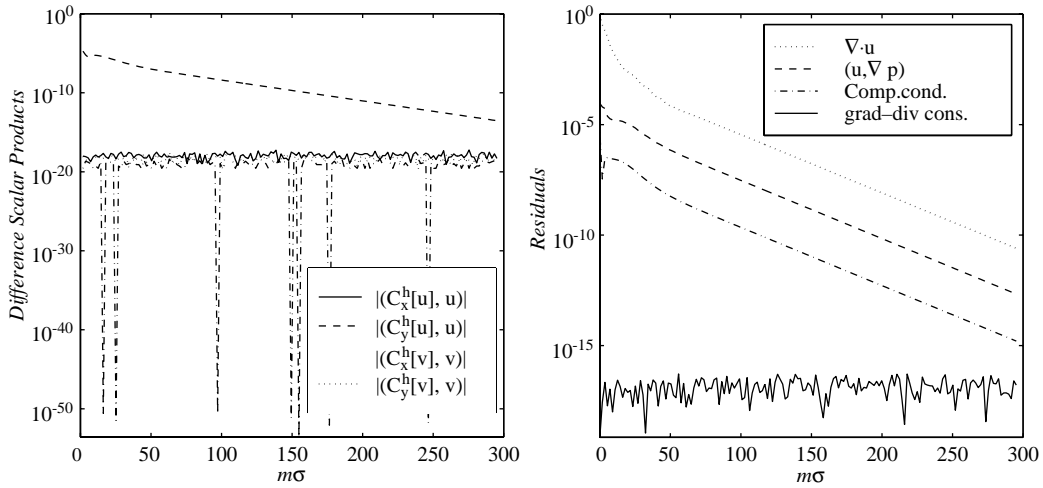


Fig. 1. Fulfillment of the requirements (i), (ii), (iv) and (v) formulated in Section 4.1 for the difference scheme defined in Section 4.2 with  $Re=1000$ ,  $h=\frac{1}{64}$ ,  $\sigma=0.09$ ,  $\tau=10^{20}$ . (a) Conserving properties of advection terms: conditions (40)–(42); (b) Residuals of Eqs. (2), (45), (32), and (33).

3D algorithm have been examined and, of course, the results are practically identical as we expected. In all tests the admissible tolerance is chosen to be  $\varepsilon \leq 10^{-10}$  for the uniform norms of residuals of the equations for velocity components and pressure. The initial guess is taken to be zero. All computations are done using double precision arithmetics.

The numerical values of the scalar products in (40)–(42) versus the number of iterations are plotted in Fig. 1(a). While Fig. 1(b) presents the residuals of continuity equation (2), equality (45) for energy conservation of pressure term, compatibility condition (32) for pressure, and consistency between gradient and divergence operators (33). Clearly, the requirements for conservation of the square of velocity in the case of homogenous boundary conditions and consistency between gradient and divergent operators are satisfied in order of the round-off error in double precision arithmetics. All the rest residuals, including those of the equations for velocity and pressure, approach zero exponentially.

As it has been already mentioned the iterative algorithm for solving the stationary Navier–Stokes problems ensures fulfillment of the continuity equation at convergence. It is known that there is one-to-one correspondence between the original Navier–Stokes equations and the formulation with skew-symmetric form of the advection term only if the continuity equation is satisfied. Hence, it is natural the residuals to approach zero in conjunction with divergence of velocity. Since we examine the stationary algorithm, only the converged solution is of interest.

Fig. 2(a) illustrates the advantage of the skew-symmetric form of nonlinear term used in the proposed splitting scheme over the divergent form. In this experiment we employ semi-implicit scheme (explicit only for the nonlinear operator). For a specific set of parameters, namely  $Re=1000$ ,  $\tau=0.2$ ,  $h=\frac{1}{64}$ , the scheme using divergent form of advection term does not converge, while the skew-symmetric scheme is convergent. Fig. 2(b) shows the convergence of the iterations for such a high Reynolds number as  $Re=10000$ . In this case it is necessary to employ fully implicit scheme

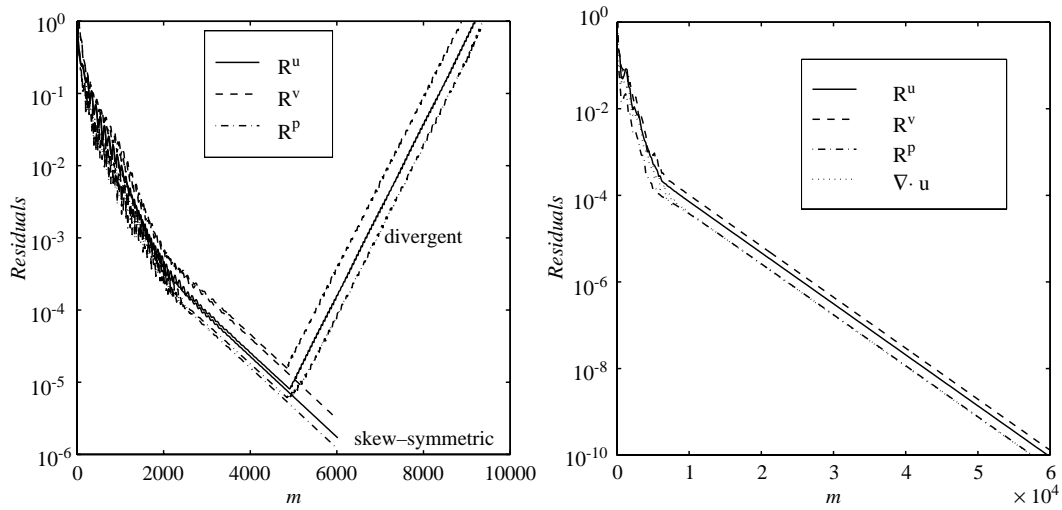


Fig. 2. Rate of convergence using different approximations of the advection term. (a) Skew-symmetric and divergent form (semi-implicit scheme),  $Re=1000$ ,  $h = \frac{1}{64}$ ,  $\sigma=0.2$ ; (b) Conservative skew-symmetric form (implicit scheme),  $Re=10,000$ ,  $h = 1/512$ ,  $\sigma = 0.05$ .

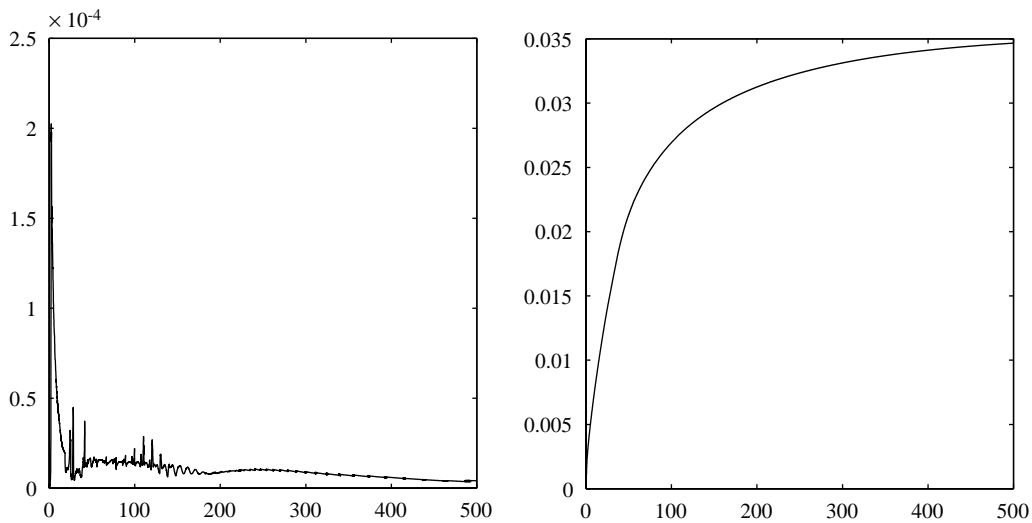


Fig. 3. Time evolution of divergence (left) and kinetic energy (right).

which enables us to achieve convergence using high resolutions. The norms of the residuals of the equations for  $u$ ,  $v$ , and  $p$ , as well as of  $\nabla \cdot \mathbf{u}$ , again approach zero exponentially.

More information about the convergence dependency on the scheme parameters such as  $Re$ ,  $\sigma$ , initial guess, grid, etc. can be found in [11].

### 5.2. Unsteady algorithm

Next the algorithm for solving the unsteady problem is investigated. We performed computation using the 2D unsteady cavity problem with  $Re = 10\,000$  started from rest, i.e., zero initial condition. A uniform grid is used with  $N_x = N_y = 257$ ,  $\tau = 0.1$ ,  $\sigma = 0.1$ ,  $0 \leq t \leq 500$ . At  $t = 500$  the flow is almost steady. The divergence of velocity  $\nabla \cdot \mathbf{u}$  and the kinetic energy  $K$  in dependence of  $t$  are presented in Fig. 3. There are some oscillations in the divergence in the beginning when the flow is highly unsteady and for  $t > 250$  the decrease of divergence is monotone. The kinetic energy is growing and also tends to stabilizing.

## 6. Conclusions

In order to conserve the properties of the original problem, difference schemes for incompressible flows should satisfy the formulated in Section 4.1 analytical requirements. We proved that some conserving properties are satisfied for the vectorial operator splitting scheme under not so restrictive assumptions. Therefore, we succeeded to achieve strong stability in solving higher Reynolds number flows, namely: lid-driven cavity problem up to  $Re = 11\,000$  for square,  $Re = 6\,000$  for deep,  $Re = 2\,000$  for cubic cavity, and up to  $Re = 1\,550$  for backward-facing step in a 2D channel. Various numerical results indicating the excellent conservation properties of the proposed scheme are presented.

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